第二章 Microeconomics HW2 (中级微观经济学作业 2)

Question 1

Consider a consumer with two goods. Let the price vector be $p = (p_1, p_2)$ and income be y. The indirect utility function is

$$v = \frac{y}{\frac{p_1}{\alpha} + \frac{p_2}{\beta}}$$

(a) Derive the expenditure function.

(b) Derive the Hicksian demand function.

(c) Derive the Marshallian demand function.

(d) For good 1, verify the Slutsky equation:

$$\frac{\partial x_1(p,y)}{\partial p_1} = \frac{\partial h_1(p,u)}{\partial p_1} - \frac{\partial x_1(p,y)}{\partial y} \cdot x_1(p,y)$$

(e) Find the corresponding direct utility function.

【详细解答】

(a) 支出函数 (Expenditure function)

利用间接效用函数与支出函数的对偶关系：$v(p, e(p,u)) = u$。将 y 替换为 $e(p,u)$，令其等于 u：

$$u = \frac{e(p,u)}{\frac{p_1}{\alpha} + \frac{p_2}{\beta}}$$

整理可得支出函数：

$$e(p,u) = u\left(\frac{p_1}{\alpha} + \frac{p_2}{\beta}\right)$$

(b) 希克斯需求函数 (Hicksian demand function)

根据谢泼德引理 (Shephard's Lemma)，$h_i(p,u)=\frac{\partial e(p,u)}{\partial p_i}$：

$$h_1(p,u)=\frac{\partial e}{\partial p_1}=\frac{u}{\alpha}$$

$$h_2(p,u)=\frac{\partial e}{\partial p_2}=\frac{u}{\beta}$$

(c) 马歇尔需求函数 (Marshallian demand function)

根据罗伊恒等式 (Roy's Identity)，$x_i(p,y)=-\frac{\partial v/\partial p_i}{\partial v/\partial y}$。已知：

$$\frac{\partial v}{\partial p_1}=-\frac{y/\alpha}{\left(\frac{p_1}{\alpha}+\frac{p_2}{\beta}\right)^2},\quad \frac{\partial v}{\partial y}=\frac{1}{\frac{p_1}{\alpha}+\frac{p_2}{\beta}}$$

代入罗伊恒等式：

$$x_1(p,y)=-\frac{-\frac{y/\alpha}{(p_1/\alpha+p_2/\beta)^2}}{\frac{1}{p_1/\alpha+p_2/\beta}}=\frac{y/\alpha}{\frac{p_1}{\alpha}+\frac{p_2}{\beta}}$$

同理可得：

$$x_2(p,y)=\frac{y/\beta}{\frac{p_1}{\alpha}+\frac{p_2}{\beta}}$$

(d) 验证斯卢茨基方程 (Verify the Slutsky equation)

左边 (价格总效应)：

$$\frac{\partial x_1}{\partial p_1}=\frac{\partial}{\partial p_1}\left[\frac{y/\alpha}{p_1/\alpha+p_2/\beta}\right]=-\frac{y/\alpha^2}{\left(\frac{p_1}{\alpha}+\frac{p_2}{\beta}\right)^2}$$

右边第一项 (替代效应)：$\frac{\partial h_1}{\partial p_1}=\frac{\partial(u/\alpha)}{\partial p_1}=0$。

右边第二项 (收入效应)：

$$\frac{\partial x_1}{\partial y}=\frac{1/\alpha}{p_1/\alpha+p_2/\beta}$$

$$-\frac{\partial x_1}{\partial y}x_1=-\left(\frac{1/\alpha}{p_1/\alpha+p_2/\beta}\right)\left(\frac{y/\alpha}{p_1/\alpha+p_2/\beta}\right)=-\frac{y/\alpha^2}{\left(\frac{p_1}{\alpha}+\frac{p_2}{\beta}\right)^2}$$

左边 = 右边，Slutsky 方程得证。

(e) 直接效用函数 (Direct utility function)

从支出函数 $e(p,u)=u\frac{p_1}{\alpha}+u\frac{p_2}{\beta}$ 可以看出，支出在价格上是线性的。这意味着商品 1 和商品 2 在消费中必须按照严格的固定比例搭配。由希克斯需求 $x_1=u/\alpha$ 和 $x_2=u/\beta$ 可知，$u=\alpha x_1=\beta x_2$。因此，对应的直接效用函数是**里昂惕夫型 (Leontief)**：

$$u(x_1,x_2)=\min\{\alpha x_1,\beta x_2\}$$

Question 2

Consider a risk-averse individual with wealth w but facing probability of losing D dollars. An insurance policy is available that pays money in the event of the loss, at a price of $\$q$ with certainty per $1 of payout. Thus, if α units of insurance are bought, the wealth of the individual will be $w - \alpha q$ if there is no loss and $w - \alpha q - D + \alpha$ if the loss occurs.

(a) Suppose $q = \pi$. What is the optimal value of α? Which state will the individual prefer?

(b) How does your answer change if $q > \pi$?

(c) Suppose once again $q = \pi$. Now suppose that the individual has a chance to buy a report that will forecast whether the loss will occur. The report has a probability p of providing the correct answer and $(1 - p)$ of giving "no answer". Assume the insurance company has a rule that nobody can insure for more than D. Derive an expression for the sure payment V the individual would be willing to pay to purchase such a report.

【详细解答】

消费者的期望效用函数为：

$$EU(\alpha) = (1 - \pi)u(w - \alpha q) + \pi u(w - \alpha q - D + \alpha)$$

最大化 EU 对 α 求导，一阶条件 (FOC) 为：

$$-q(1 - \pi)u'(W_{NL}) + (1 - q)\pi u'(W_L) = 0 \implies \frac{u'(W_L)}{u'(W_{NL})} = \frac{q(1 - \pi)}{\pi(1 - q)}$$

其中 $W_{NL} = w - \alpha q$，$W_L = w - \alpha q - D + \alpha$。

(a) 当 $q = \pi$ (公平保费 Fair Insurance)： 代入 FOC 可得 $\frac{u'(W_L)}{u'(W_{NL})} = 1 \implies u'(W_L) = u'(W_{NL})$。由于个体风险规避 ($u'' < 0$)，必有 $W_L = W_{NL}$。即：$w - \alpha\pi - D + \alpha = w - \alpha\pi \implies$ **ff = D**。最优选择是**全额保险 (Full insurance)**。由于两种状态下财富完全相等，个体对“是否发生损失”完全**无差异 (Indifferent)**，不偏好任何一种状态。

(b) 当 $q > \pi$ (不公平保费 Unfair Insurance)： 此时 $\frac{q}{1-q} > \frac{\pi}{1-\pi}$，代入 FOC 得到 $\frac{u'(W_L)}{u'(W_{NL})} > 1$。由于边际效用递减，这意味着 $W_L < W_{NL}$。即：$w - \alpha q - D + \alpha < w - \alpha q \implies$ **ff < D**。此时最优选择变为**部分保险 (Partial insurance)**。个体更偏好“不发生损失”的状态。

(c) 预测报告的价值： 如果没有报告，由于 $q=\pi$，个体购买 $\alpha=D$，期望效用为 $EU_{no\_report}=u(w-\pi D)$。
如果有报告并支付了确定的费用 V：

- 概率 $p\pi$:报告正确预测"**会发生损失**"。个体知道必发生损失,希望买尽量多的保险。受限于最高保额,买 $\alpha=D$。最终财富 $=w-\pi D-D+D-V=w-\pi D-V$。
- 概率 $p(1-\pi)$:报告正确预测"**不会发生损失**"。个体完全不买保险 $(\alpha=0)$。最终财富 $=w-V$。
- 概率 $1-p$：报告给出"**没有答案**"。个体面临与原先相同的概率 π，因此依然购买全额保险 $\alpha=D$。最终财富 $=w-\pi D-V$。

购买报告的期望效用为：

$$EU_{report}=p\pi\cdot u(w-\pi D-V)+p(1-\pi)\cdot u(w-V)+(1-p)\cdot u(w-\pi D-V)$$

合并同类项得到：

$$EU_{report}=p(1-\pi)u(w-V)+(1-p+p\pi)u(w-\pi D-V)$$

意愿支付的最高费用 V 必须满足 $EU_{report}=EU_{no\_report}$，因此隐式表达式为：

$$\mathbf{p(1-\text{ß})u(w-V)+(1-p+p\text{ß})u(w-\text{ß}D-V)=u(w-\text{ß}D)}$$

Question 3

A firm has production function $f(z_1,z_2)=z_1^{1/2}(z_2-1)^{1/2}$. The prices of the inputs are r_1 and r_2.
(a) Find MP_1, MP_2, and MRTS.
(b) If z_2 is fixed at 5, what is the short-run cost function? Find the short-run marginal cost and average cost.
(c) What are the long-run input demand functions? What is the long-run cost function? Find the long-run marginal cost and average cost.
(d) Does the production function exhibit increasing, constant or decreasing returns to scale.

【详细解答】

(a) 边际产量与边际技术替代率：

$$MP_1 = \frac{\partial f}{\partial z_1} = \frac{1}{2} z_1^{-1/2} (z_2 - 1)^{1/2}$$

$$MP_2 = \frac{\partial f}{\partial z_2} = \frac{1}{2} z_1^{1/2} (z_2 - 1)^{-1/2}$$

$$MRTS = \frac{MP_1}{MP_2} = \frac{z_2 - 1}{z_1}$$

(b) 短期成本函数 ($z_2 = 5$)：产量 $q = z_1^{1/2}(5-1)^{1/2} = 2z_1^{1/2} \implies z_1 = \frac{q^2}{4}$。短期总成本：$SRTC(q) = r_1 z_1 + r_2 z_2 = \mathbf{\frac{r_1 q^2}{4} + 5r_2}$。短期边际成本：$SRMC = \frac{\partial SRTC}{\partial q} = \mathbf{\frac{r_1 q}{2}}$。短期平均成本：$SRAC = \frac{SRTC}{q} = \mathbf{\frac{r_1 q}{4} + \frac{5r_2}{q}}$。

(c) 长期成本与投入需求：成本最小化一阶条件：$MRTS = \frac{r_1}{r_2} \implies \frac{z_2 - 1}{z_1} = \frac{r_1}{r_2} \implies z_1 = \frac{r_2}{r_1}(z_2 - 1)$。代入生产函数：

$$q = \left[\frac{r_2}{r_1}(z_2 - 1)^2\right]^{1/2} = \left(\frac{r_2}{r_1}\right)^{1/2} (z_2 - 1)$$

解得**长期投入需求函数**：

$$z_2(q, r) = q\sqrt{\frac{r_1}{r_2}} + 1, \quad z_1(q, r) = q\sqrt{\frac{r_2}{r_1}}$$

长期总成本：

$$LRTC(q) = r_1 z_1 + r_2 z_2 = r_1 q \sqrt{\frac{r_2}{r_1}} + r_2 \left(q\sqrt{\frac{r_1}{r_2}} + 1\right) = \mathbf{2q\sqrt{r_1 r_2} + r_2}$$

长期边际成本：$LRMC = \frac{\partial LRTC}{\partial q} = \mathbf{2\sqrt{r_1 r_2}}$。

长期平均成本：$LRAC = \frac{LRTC}{q} = \mathbf{2\sqrt{r_1 r_2} + \frac{r_2}{q}}$。

(d) 规模报酬 (Returns to Scale)：设比例因子 $t > 1$：

$$f(tz_1, tz_2) = (tz_1)^{1/2}(tz_2 - 1)^{1/2} = t z_1^{1/2}\left(z_2 - \frac{1}{t}\right)^{1/2}$$

由于 $t > 1$，则 $\frac{1}{t} < 1$，所以 $z_2 - \frac{1}{t} > z_2 - 1$。因此：$f(tz_1, tz_2) > t \cdot z_1^{1/2}(z_2 - 1)^{1/2} = t \cdot f(z_1, z_2)$。这表明生产函数具有**规模报酬递增 (Increasing returns to scale)**。*(注：从 $LRAC$ 随产量 q 增加而单调递减也能直接判断出规模报酬递增。)*

Question 4

The value of non-market goods. Consider a farmer whose production function is $y = f(x, q)$, where x is a vector of market inputs and q is a non-market quality variable (e.g. soil fertility, air quality). The price of output is fixed at p and the prices of the market inputs are w.

(a) Derive a simple expression for the marginal profit of the non-market good q.

(b) Derive an alternative expression for the value of q based on a market input serving as a substitute for q (Hint: the expression here should be related to MRTS).

(c) What is the economic interpretation for your results in part (b)?

【详细解答】

(a) 非市场物品的边际利润表达式： 农民的利润最大化问题为：$\pi(p, w, q) = \max_x\{pf(x, q) - wx\}$。根据**包络定理 (Envelope Theorem)**，利润函数对参数 q 的导数等于目标函数对其偏导：

$$\frac{\partial \pi}{\partial q} = p\frac{\partial f(x^*, q)}{\partial q} = \mathbf{p \cdot MP_q}$$

这表明，环境质量 q 的边际价值等于它直接增加的产量的市场价值。

(b) 基于市场替代投入的替代表达式： 在利润最大化的一阶条件中，对任意市场投入 x_i，有 $p\frac{\partial f}{\partial x_i} = w_i \implies p = \frac{w_i}{MP_{x_i}}$。将此代入 (a) 的结果中：

$$\frac{\partial \pi}{\partial q} = \left(\frac{w_i}{MP_{x_i}}\right) MP_q = w_i \left(\frac{MP_q}{MP_{x_i}}\right)$$

由于 $\frac{MP_q}{MP_{x_i}}$ 正是非市场物品 q 与市场投入 x_i 之间的**边际技术替代率 (MRTS)**，因此：

$$\frac{\partial \text{ß}}{\partial \mathbf{q}} = \mathbf{w_i \cdot MRTS_{q,x_i}}$$

(c) 经济学直觉解释 (Economic interpretation)： 公式 $\frac{\partial \pi}{\partial q} = w_i \cdot MRTS_{q,x_i}$ 体现了著名的“**重置成本 / 替代成本 (Replacement Cost)**”估值法。它的经济含义是：非市场环境商品 (如自然土壤肥力 q) 的价值，正好等于它**所能节省的市场替代投入品的成本**。例如，如果天然土壤肥力 q 提高了一点，农民在保持产量不变的情况下，可以少买一些人造化肥 (x_i)。那么，这部分天然肥力的“影子价格”，就正好等于农民由此省下的购买化肥的钱 ($w_i\times$ 节省的化肥数量)。